

## Derivatives

### Chain Rule

1. Find the derivative of  $f(x) = \cos^3\left(\frac{1}{3x+1}\right)$ .

**Solution:** We use the chain rule to get that

$$\begin{aligned}\frac{df}{dx}(x) &= 3 \cos^2\left(\frac{1}{3x+1}\right) \cdot \frac{d}{dx} \cos\left(\frac{1}{3x+1}\right) \\ &= 3 \cos^2\left(\frac{1}{3x+1}\right) \cdot \left[-\sin\left(\frac{1}{3x+1}\right) \cdot \frac{d}{dx} \frac{1}{3x+1}\right] \\ &= 3 \cos^2\left(\frac{1}{3x+1}\right) \cdot \left[-\sin\left(\frac{1}{3x+1}\right)\right] \cdot \frac{-1}{(3x+1)^2} \cdot \frac{d}{dx} 3x \\ &= 3 \cos^2\left(\frac{1}{3x+1}\right) \cdot \sin\left(\frac{1}{3x+1}\right) \cdot \frac{3}{(3x+1)^2}.\end{aligned}$$

2. Find the derivative of  $f(x) = x^5 \cos(3x)$ .

**Solution:** We use the product rule to get that the derivative is

$$\begin{aligned}x^5 \cdot \frac{d}{dx} \cos(3x) + \cos(3x) \cdot \frac{d}{dx} x^5 &= x^5 \cdot -\sin(3x) \cdot 3 + \cos(3x) \cdot 5x^4 \\ &= -3x^5 \sin(3x) + 5x^4 \cos(3x).\end{aligned}$$

3. Find the derivative of  $\sqrt{\sin(2x)}$ .

**Solution:** Using the chain rule, this is

$$\frac{d}{dx} (\sin(2x))^{1/2} = \frac{1}{2} \cdot (\sin(2x))^{-1/2} \cdot \cos(2x) \cdot 2 = \frac{\cos(2x)}{\sqrt{\sin(2x)}}.$$

4. Find the derivative of  $\sin(\sqrt{x})$ .

**Solution:** Again we use the chain rule to get

$$\frac{d}{dx} \sin(x^{1/2}) = \cos(x^{1/2}) \cdot \frac{1}{2} \cdot x^{-1/2} = \frac{\cos(\sqrt{x})}{2\sqrt{x}}.$$

5. Find the derivative of  $\cot(3x^2)$ .

**Solution:** Chain rule gives

$$\frac{d}{dx} \cot(3x^2) = -\csc^2(3x^2) \cdot 6x = -6x \csc^2(3x^2).$$

## Inverse Functions

6. Find the derivative of  $\operatorname{arccot}(x)$ .

**Solution:** We know that  $\cot(\operatorname{arccot}(x)) = x$  by definition and so taking the derivative with respect to  $x$  of both sides of the equation gives

$$-\csc^2(\operatorname{arccot}(x)) \cdot \operatorname{arccot}'(x) = 1,$$

and hence the derivative of  $\operatorname{arccot}(x)$  is  $-(\csc(\operatorname{arccot}(x)))^{-2}$ . We want to get rid of the  $\csc(\operatorname{arccot}(x))$  part and get it terms of a polynomial of  $x$ . In order to do so, we use the “Pythagorean Theorem” of trigonometry, which is  $\sin^2(x) + \cos^2(x) = 1$ . Dividing both sides by  $\sin^2(x)$  gives  $1 + \cot^2(x) = \csc^2(x)$  and hence

$$\frac{-1}{\csc^2(\operatorname{arccot}(x))} = \frac{-1}{1 + [\cot(\operatorname{arccot}(x))]^2} = \frac{-1}{1 + x^2}.$$

7. Find the derivative of  $\arcsin(x)$ .

**Solution:** Starting with  $\sin(\arcsin(x)) = x$ , we take the derivative to get

$$\cos(\arcsin(x)) \cdot \arcsin'(x) = 1,$$

and hence the derivative is  $\frac{1}{\cos(\arcsin(x))}$ . Now we use the fact that  $\sin^2(x) + \cos^2(x) = 1$ , so that  $\cos(x) = \sqrt{1 - \sin^2(x)}$ , to get that the derivative is

$$\frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1 - x^2}}.$$

8. Let  $f(x) = xe^x$  and let  $g$  be the inverse function of  $f$ . Given that  $f(1) = e$ , find  $g'(e)$ .

**Solution:** Since  $g$  is the inverse of  $f$ , we know that  $f(g(x)) = x$  and thus taking the derivative gives  $f'(g(x)) \cdot g'(x) = 1$ . We plug in  $x = e$  to get  $f'(g(e)) \cdot g'(e) = 1$ . Since  $f(1) = e$  and  $g$  is the inverse to  $f$ , we know that  $g(f(1)) = 1$ , and hence  $g(e) = 1$ . Now calculate that  $f'(x) = x \cdot \frac{d}{dx}e^x + e^x \cdot \frac{d}{dx}x = xe^x + e^x$ . Therefore, we have that

$$g'(e) = \frac{1}{f'(g(e))} = \frac{1}{f'(1)} = \frac{1}{1 \cdot e^1 + e^1} = \frac{1}{2e}.$$

9. Let  $f(x) = x^5 + 4x^3$  and let  $g$  be the inverse function of  $f$ . Given that  $f(1) = 5$ , find  $g'(5)$ .

**Solution:** Since  $f(1) = 5$ , we know that  $g(5) = 1$ . Also, note that  $f'(x) = 5x^4 + 12x^2$ . So, we have that

$$g'(5) = \frac{1}{f'(g(5))} = \frac{1}{f'(1)} = \frac{1}{5 + 12} = \frac{1}{17}.$$

10. Let  $f(x) = x^5 + x + 1$  and let  $g$  be the inverse function of  $f$ . Find the derivative of  $g$  at  $(3, 1)$ .

**Solution:** Note that  $f'(x) = 5x^4 + 1$ . We have that

$$g'(x) = \frac{1}{f'(y)} \implies g'(3) = \frac{1}{f'(1)} = \frac{1}{5 + 1} = \frac{1}{6}.$$

## Implicit Differentiation

11. Find the derivative of the function  $y^2(6 - x) = x^3$  at  $(3, 3)$ .

**Solution:** We take the derivative of both sides with respect to  $x$  and use implicit differentiation to get that

$$\begin{aligned}(6-x)\frac{d}{dx}y^2 + y^2 \cdot \frac{d}{dx}(6-x) &= \frac{d}{dx}x^3 \\ \implies (6-x)(2y) \cdot \frac{dy}{dx} + y^2(-1) &= 3x^2 \\ \implies \frac{dy}{dx} &= \frac{3x^2 + y^2}{2y(6-x)}.\end{aligned}$$

We want to find the derivative at the point  $(3, 3)$ , so when  $x = y = 3$ . Plugging that into the equation, we get that value

$$\frac{dy}{dx} = \frac{3(3^2) + 3^2}{2(3)(6-3)} = \frac{36}{18} = 2.$$

12. Find  $\frac{dy}{dx}$  given that  $\frac{1}{y} + \frac{1}{x^2} = 1$ .

**Solution:** We take the derivative to get that

$$-y^{-2} \cdot y' + (-2x^{-3}) = 0 \implies y' = \frac{2x^{-3}}{-y^{-2}} = \frac{-2y^2}{x^3}.$$

13. Let  $y^2 = x^2(x-1)$ . At what points is  $\frac{dy}{dx}$  not defined?

**Solution:** We use implicit differentiation to get that

$$2y\frac{dy}{dx} = 3x^2 - 2x \implies \frac{dy}{dx} = \frac{3x^2 - 2x}{2y}.$$

Thus, the derivative is not defined whenever  $y = 0$ . So, we need to find all points on the curve such that  $y = 0$ . Plugging that in gives  $x^2(x-1) = 0$  and hence  $x = 0, 1$  are the solutions. So, the derivative is not defined at the points  $(0, 0)$  and  $(1, 0)$ .

14. Find  $\frac{dy}{dx}$  if  $\ln(xy) = e^y$ .

**Solution:** Taking the derivative of both sides gives

$$\frac{1}{xy} \cdot (x \cdot y' + y \cdot 1) = e^y \cdot y'.$$

So, we have that

$$y'e^y - \frac{y'}{y} = \frac{1}{x},$$

and so

$$\frac{dy}{dx} = \frac{1}{x(e^y - 1/y)}.$$

15. Find when the curve  $x^4 = 2x^2 - y^2$  has a horizontal derivative.

**Solution:** Taking the derivative gives  $4x^3 = 4x - 2yy'$  and hence

$$\frac{dy}{dx} = \frac{4x - 4x^3}{2y} = \frac{2x - 2x^3}{y}.$$

When the derivative is equal to 0, we have that  $2x - 2x^3 = 0$  and hence  $2x(1 - x^2) = 0$  so  $x = 0$  or  $x = \pm 1$ . Plugging  $x = -1$  into the original equation gives  $1 = 2 - y^2$  so  $y = \pm 1$  and plugging  $x = 1$  into the original equation gives  $y = \pm 1$  as well. Finally, plugging  $x = 0$  into the original equation gives  $y = 0$ . But, note that at this point, the derivative is not defined since  $y$  is in the denominator. Solving for  $y$  as  $y = \sqrt{2x^2 - x^4}$  gives

$$\frac{dy}{dx} = \frac{2x - 2x^3}{\sqrt{2x^2 - x^4}} = \frac{2x - 2x^3}{x\sqrt{2 - x^2}} = \frac{2 - 2x^2}{\sqrt{2 - x^2}},$$

and by doing this, we see that the slope at  $(0, 0)$  is not 0. Thus, there are only 4 points where the derivative is 0, namely  $\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$ .

## L'Hopital's Rule

16. Find  $\lim_{x \rightarrow \infty} \sqrt{2x+1} - \sqrt{x+1}$ .

**Solution:** Plugging in  $x = \infty$  gives  $\infty - \infty$ , which is an indeterminate. We multiply by the conjugate as our first problem solving technique to get

$$\lim_{x \rightarrow \infty} \sqrt{2x+1} - \sqrt{x+1} = \lim_{x \rightarrow \infty} \sqrt{2x+1} - \sqrt{x+1} \cdot \frac{\sqrt{2x+1} + \sqrt{x+1}}{\sqrt{2x+1} + \sqrt{x+1}}$$

$$= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{2x+1} + \sqrt{x+1}}.$$

Plugging in  $x = \infty$  give  $\frac{\infty}{\infty}$  which again is an indeterminate. But, this time, we can use L'Hopital's rule and take the derivative of the top and bottom to get that this integral is equal to

$$\lim_{x \rightarrow \infty} \frac{1}{(2x+1)^{-1/2} + (2\sqrt{x+1})^{-1}}.$$

Now plugging in  $x = \infty$  gives

$$\frac{1}{1/\infty + 1/\infty} = \frac{1}{0^+ + 0^+} = \frac{1}{0^+} = \infty.$$

Thus, the original limit is equal to  $\infty$ .

17. Find  $\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 1}}$ .

**Solution:** Plugging in  $\infty$  gives  $\infty/\infty$  which is an indeterminate and hence we try to use L'Hopital's rule. Doing so gives

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 1}} = \lim_{x \rightarrow \infty} \frac{1}{\frac{2x}{2\sqrt{x^2 - 1}}} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 - 1}}{x}.$$

Plugging in  $\infty$  and using L'Hopital's rule again gives

$$= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 1}},$$

so it leads us back to where we originally were. We need another technique to solve this. Since this is an infinite limit with  $x \rightarrow \pm\infty$ , we can divide the top and bottom by the largest power of  $x$  that we see, which is just  $x$ . Doing so gives

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 - 1}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 - 1/x^2}} = \frac{1}{\sqrt{1 - 1/\infty}} = \frac{1}{1} = 1.$$

So, sometimes we need to use other techniques.

18. Find  $\lim_{x \rightarrow \infty} x^{1/x}$ .

**Solution:** Plugging in  $\infty$  gives  $\infty^0$ , which is an indeterminate. In order to get rid of an exponential indeterminate, we can raise everything to the  $e$ th power. So

$$\lim_{x \rightarrow \infty} x^{1/x} = \lim_{x \rightarrow \infty} (e^{\ln x})^{1/x} = \lim_{x \rightarrow \infty} e^{(\ln x)/x}.$$

It suffices to compute the limit of  $(\ln x)/x$  as  $x \rightarrow \infty$ . Plugging in  $\infty$  gives  $\infty/\infty$  which is an indeterminate and we can use L'Hopital's rule to get that

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = \frac{1}{\infty} = 0.$$

Thus, the original limit was  $e^0 = 1$ .

19. Find  $\lim_{x \rightarrow 0^+} x^x$ .

**Solution:** Plugging in  $0^+$  gives  $0^0$ , which is an exponential indeterminate. Thus, we have that

$$\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{x \ln x}.$$

So, we need to find the derivative of  $x \ln x = \ln x/(1/x)$ . Plugging in  $0^+$  gives the indeterminate  $-\infty/\infty$ . Thus, we can use L'Hopital's rule to get

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} -x = 0.$$

Thus, the original limit was  $e^0 = 1$ .

20. Find  $\lim_{x \rightarrow 0} \frac{\cos(x) - 1 + \sin(x)^2/2}{x^4}$ .

**Solution:** We repeatedly use L'Hopital's rule to get

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos(x) - 1 + \sin(x)^2/2}{x^4} &= \lim_{x \rightarrow 0} \frac{-\sin(x) + 2 \sin(x) \cos(x)/2}{4x^3} \\ &= \lim_{x \rightarrow 0} \frac{-\cos(x) + \sin(x)(-\sin(x)) + \cos(x) \cos(x)}{12x^2} \\ &= \lim_{x \rightarrow 0} \frac{-\cos(x) + \cos^2(x) - \sin^2(x)}{12x^2} = \lim_{x \rightarrow 0} \frac{\sin(x) - 2 \cos(x) \sin(x) - 2 \sin(x) \cos(x)}{24x} \\ &= \lim_{x \rightarrow 0} \frac{\sin(x) - 4 \sin(x) \cos(x)}{24x} \\ &= \lim_{x \rightarrow 0} \frac{\cos(x) - 4 \sin(x)(-\sin(x)) - 4(\cos(x)) \cos(x)}{24} = \frac{1 - 4}{24} = \frac{-1}{8}. \end{aligned}$$

## Application

### Optimization

21. Find the area of the smallest triangle formed by the  $x$  axis,  $y$  axis, and a line that goes through the point  $(1, 1)$ .

**Solution:** Suppose that the line goes through the point  $(0, y_0)$ . Then, the slope of the line is  $\frac{1-y_0}{1}$  and is described by the line  $y - y_0 = \frac{1-y_0}{1}x$ . The  $x$  intercept is when  $y = 0$  or when  $x = \frac{y_0}{y_0-1}$ . Thus, the area of the triangle is

$$A(y_0) = \frac{1}{2} \cdot y_0 \cdot \frac{y_0}{y_0 - 1} = \frac{y_0^2}{2y_0 - 2}.$$

Setting the derivative equal to zero gives  $A'(y) = \frac{y^2 - 2y(y-1)}{2(y-1)^2} = 0$  so  $y^2 - 2y^2 + 2y = y(2 - y) = 0$ . The two solutions are  $y = 0$  and  $y = 2$ . The second derivative is  $\frac{-1}{(y-1)^3}$  and so  $y_0 = 0$  is a local minimum and  $y_0 = 2$  is a local maximum. So the maximum area is achieved by the line that goes through  $(1, 1)$  and  $(0, 2)$  with area  $\frac{2 \cdot 2}{2} = 2$ .

22. Find the largest rectangle that can be inscribed into a semicircle of radius 2 so that one side of the rectangle is part of the diameter of the semicircle.

**Solution:** Let the height of the rectangle be  $h$ . Then the other side of the rectangle must be  $2\sqrt{4 - h^2}$ . So we want to maximize  $2h\sqrt{4 - h^2}$ , which is the same as maximizing its square  $4h^2(4 - h^2)$ . Setting the derivative equal to 0 gives  $32h - 16h^3 = 0$  so  $h = \sqrt{2}$ . The area is  $2\sqrt{2} \cdot \sqrt{2} = 4$ .

23. Find the point on the curve  $y = 1 - \sqrt{x}$  closest to  $(1, 1)$ .

**Solution:** The equation of distance is

$$\begin{aligned} \sqrt{(x - x_0)^2 + (y - y_0)^2} &= \sqrt{(x - 1)^2 + (1 - \sqrt{x} - 1)^2} = \sqrt{x^2 - 2x + 1 + x} \\ &= \sqrt{x^2 - x + 1}. \end{aligned}$$

Thus, we want to find when  $\sqrt{x^2 - x + 1}$  is minimized. In order to find the minimum, we can take the derivative and set it equal to 0. Doing so gives

$$\frac{2x - 1}{2\sqrt{x^2 - x + 1}} = 0 \implies x = \frac{1}{2}.$$

So, the point is  $(1/2, 1 - \sqrt{1/2}) = (1/2, 1 - 1/\sqrt{2})$ .



24. A rectangle is inscribed under the curve  $\sin x$  for  $0 \leq x \leq \pi$ . This rectangle has two vertices on the curve and one side on the  $x$  axis. What is the maximum possible area of such a rectangle.

**Solution:** If one vertex on the  $x$  axis is at  $(x_0, 0)$ , then the other is at a point  $x'$  such that  $\sin(x_0) = \sin(x')$ . You can show that this point is  $x' = \pi - x_0$ . Thus, the vertices of this rectangle are  $(x_0, 0)$ ,  $(x_0, \sin(x_0))$ ,  $(\pi - x_0, 0)$ ,  $(\pi - x_0, \sin(x_0))$ . Thus, the area of the rectangle is  $(\pi - x_0 - x_0) \sin(x_0) = (\pi - 2x_0) \sin(x_0)$ . Taking the derivative with respect to  $x_0$  gives  $(\pi - 2x_0) \cos(x_0) + (-2) \sin(x_0) = 0$  or  $(\pi - 2x_0) \cos(x_0) = 2 \sin(x_0)$ . Solving gives  $x_0 \approx 0.7105$  so the total area is  $(\pi - 2(0.7105)) \sin(0.7105) \approx 1.122$ .

25. What is the point on  $y = e^x$  closest to  $(1, 0)$ ?

**Solution:** We want to minimize the distance, which is

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = \sqrt{(x - 1)^2 + (e^x - 0)^2} = \sqrt{(x - 1)^2 + e^{2x}}.$$

We take the derivative and set it equal to 0 to get that

$$\frac{2(x - 1) + 2e^{2x}}{2\sqrt{(x - 1)^2 + e^{2x}}} = 0 \implies 2(x - 1) + 2e^{2x} = 0.$$

The solution is  $x = 0$  since  $2(-1) + 2(1) = 0$ . Thus, the closest point is  $(0, e^0) = (0, 1)$ .

## Related Rates

26. A ball of light is bobbing up and down and whose position is given at a time  $t$  by  $4 + \sin(2t)$ . A man who is  $2m$  tall is standing  $10m$  away. How fast is the length of his shadow changing when  $t = 0$ ?

**Solution:** If the ball is at height  $d$ , then drawing a picture tells us that the height of the shadow satisfies the relation that  $\frac{h}{2} = \frac{h+10}{d}$  so  $dh = 2h + 20$ . Taking the derivative of both sides gives  $d'h + dh' = 2h'$ . We have that  $d' = 2 \cos(2t)$  and when  $t = 0$ , we have that  $d = 4 + \sin(0) = 4$  and  $d' = 2 \cos(0) = 2$ . Solving for  $h$  gives  $h = 10$  so

$$10 \cdot 2 + 4h' = 2h' \implies h' = -10.$$

So the shadow is shortening at  $10m/s$ .

27. A conical cup that is  $6cm$  wide at the top and  $5cm$  tall is filled with water is punctured at the bottom and water is coming out at a rate of  $10^{-6}\pi m^3/s$ . Initially, the cup is filled. How fast is the height of the water changing when the height is  $3cm$ ?

**Solution:** If the height of the water is  $h$ , then the radius of the cone formed by the water would be  $3/5h$  and so the volume of the water cone is  $V = \pi/3(3/5h)^2 \cdot h = \frac{3\pi h^3}{25}$ . Taking the derivative of both sides gives

$$V' = \frac{9\pi h^2 h'}{25}$$

and plugging in  $-10^{-6}\pi$  for  $V'$  and  $3 \cdot 10^{-2}$  for  $h$  gives

$$-10^{-6}\pi = \frac{9\pi \cdot 10^{-4}h'}{25} \implies h' = \frac{1}{81 \cdot 4} = \frac{1}{324} \text{ cm/s.}$$

28. Sand is being dumped in a conical pile whose radius and height always remain the same. If the sand is being dumped in at a rate of  $2\pi m^3/hr$ , how fast is the height of the sand changing when the pile is  $5cm$  tall?

**Solution:** Let the height of the pile be  $h$ . Then the radius of the pile is  $r = h$  and the volume of the pile is  $V = \frac{\pi r^2 h}{3} = \frac{\pi h^3}{3}$ . Taking the derivative gives  $V' = \pi h^2 h'$ . Now we plug in  $2\pi$  for  $V'$  and  $5 \cdot 10^{-2}$  for  $h$  to get  $h' = 800m/hr = \frac{800}{3600}m/s = \frac{2}{9}m/s$ .

29. A kite is flying at a current altitude of  $100m$ . The kite slowly flies further and further away as the string length increases at a rate of  $2cm/s$ . Assuming the altitude does not change, how fast horizontally is the kite moving when the angle the string forms with the ground is  $\pi/4$ ?

**Solution:** Let  $\ell$  be the length of the rope, and  $d$  how far horizontally the kite is flying. Then  $\ell^2 = 100^2 + d^2$ . Taking the derivative gives  $2\ell\ell' = 2dd'$ . When the angle the string forms with the ground is  $\pi/4$ , we calculate that  $\ell = 100\sqrt{2}$  and  $d = 100$  so  $d' = \frac{100\sqrt{2} \cdot 2 \cdot 10^{-2}}{100} = 2\sqrt{2} \cdot 10^{-2}m/s$  or  $2\sqrt{2}cm/s$ .

30. A ladder  $13m$  tall is lying against a wall. The bottom of the ladder is pulled out at a rate of  $10cm/s$ . How fast is the area of the triangle formed by the ladder, wall, and floor changing when the bottom of the ladder is  $5m$  away from the wall?

**Solution:** Let  $d$  be how far the bottom of the ladder is away from wall. Then the area of the triangle formed is  $\frac{1}{2} \cdot d \cdot \sqrt{169 - d^2} = A$ . Squaring both sides gives

$4A^2 = d^2(169 - d^2)$ . Now we can take the derivative to get that  $8AA' = 2dd'(169 - d^2) + d^2(-2dd')$ . When  $d = 5$ , the area is  $\frac{1}{2} \cdot 5 \cdot 12 = 30$  and so

$$8 \cdot 30 \cdot A' = 2 \cdot 5 \cdot d'(144) + 25(-10d') \implies 240A' = 1190d'.$$

Since  $d' = 10^{-1}m/s$ , we have that  $A' = \frac{119}{240}m^2/s$ .

## Taylor Series

31. Use the third order approximation to find  $\sin(0.5)$ .

**Solution:** We expand around 0 since  $\sin 0 = 0$ . We find that

$$\sin x \approx x - \frac{x^3}{6},$$

and so  $\sin(0.5) \approx 0.5 - 0.5^3/6 \approx 0.4792$ .

32. Approximate  $\sqrt{99}$  using a quadratic regression.

**Solution:** We expand  $f(x) = \sqrt{x}$  around  $x = 100$  to get that

$$\sqrt{x} \approx f(100) + f'(100)(x - 100) + \frac{f''(100)}{2}(x - 100)^2 = 10 + \frac{x - 100}{20} - \frac{(x - 100)^2}{8000}.$$

Thus, we have that

$$\sqrt{99} \approx 10 + \frac{-1}{20} - \frac{(-1)^2}{8000} = 10 - \frac{1}{20} - \frac{1}{8000}.$$

33. Use the second order approximation to find  $\ln 1.01$ .

**Solution:** We know that  $\ln 1 = 0$ . So we can expand out at  $x = 1$  to get

$$\ln x \approx 0 + (x - 1) - \frac{(x - 1)^2}{2}.$$

Thus, we have that  $\ln 1.01 \approx (0.01) - \frac{0.01^2}{2} = 0.00995$ .

34. Use the second order approximation to  $\sqrt[3]{8.1}$ .

**Solution:** A close cube that we know is  $2^3 = 8$ . So we calculate the second order Taylor series expanded at  $x = 8$  to get

$$\sqrt[3]{x} \approx 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}.$$

So plugging in 8.1 gives

$$\sqrt[3]{8.1} \approx 2 + \frac{.1}{12} - \frac{.1}{288} \approx 2.008.$$

35. Use the quintic order approximation to find  $e$ .

**Solution:** We know that  $e^0 = 1$  so we can expand  $e^x$  around  $x = 0$ . Doing so gives

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}.$$

Thus, we have that  $e^1 \approx 1 + 1 + \frac{1}{2} = \frac{1}{6} + \frac{1}{24} + \frac{1}{120} = 2.717$ .

## Newton's Method

36. Use Newton's method once to approximate  $\sqrt[3]{8.1}$ .

**Solution:** We want to find the root of the equation  $x^3 - 8.1$ . Newton's method tells us to recursively apply the equation

$$x' = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 8.1}{3x^2}.$$

Our first guess is  $x = 2$  and the next step is

$$2 - \frac{8 - 8.1}{3 \cdot 2^2} = 2 + \frac{1}{120} = 2.008.$$

37. Approximate  $\sqrt{99}$  using Newton's method once.

**Solution:** We want to find the root of the equation  $x^2 - 99$ . Newton's method tells us to recursively apply the equation

$$x' = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 99}{2x}.$$

Our first guess is  $x = 10$  and the next step is

$$10 - \frac{100 - 99}{20} = 10 - \frac{1}{20} = 9.95.$$

38. Find the critical points of  $g(x) = \sin(x) - x^2$

**Solution:** We want to find when the derivative is 0 or when  $f(x) = \cos(x) - 2x = 0$ . Taking the derivative again, we find that it is  $-\sin(x) - 2 < 0$  for all  $x$ . So this function is always decreasing and has a unique root. We plug in  $x = 0$  to start, then calculate

$$x' = x - \frac{f(x)}{f'(x)} = -\frac{1}{-2} = \frac{1}{2}.$$

The real solution is  $\approx 0.45$ .

39. Find the unique solution to  $(\pi - 2x) \cos(x) = 2 \sin(x)$  on the interval  $[0, 1]$  using Newton's method with an initial guess of  $x = \frac{\pi}{4}$ .

**Solution:** We want to find the root of  $f(x) = (\pi - 2x) \cos x - 2 \sin x = 0$ . Taking the derivative, we have that  $f'(x) = -(\pi - 2x) \sin x - 4 \cos x$ . Now using Newton's method gets that

$$x' = x - \frac{f(x)}{f'(x)} = \frac{\pi}{4} - \frac{(\pi/2)(\sqrt{2}/2) - 2(\sqrt{2}/2)}{-(\pi/2)(\sqrt{2}/2) - 4(\sqrt{2}/2)} = \frac{\pi}{4} - \frac{\pi\sqrt{2} - 4\sqrt{2}}{-\pi\sqrt{2} - 8\sqrt{2}} \approx \approx 0.7084.$$

40. Find when  $\cos(x) = x$  using Newton's method and an initial guess of  $x = \frac{\pi}{6}$ .

**Solution:** We want to find the root of  $f(x) = \cos(x) - x = 0$ . We calculate that  $f'(x) = -\sin(x) - 1$ . Newton's method gives us that

$$x' = x - \frac{f(x)}{f'(x)} = \frac{\pi}{6} - \frac{\sqrt{3}/2 - \pi/6}{-1/2 - 1} = \frac{\pi}{6} + \sqrt{33} - \frac{\pi}{9} = \frac{\pi}{18} + \sqrt{33} \approx 0.7519.$$

# Functions

## Domain/Range

41. Find the domain of  $y = \sqrt{9 - (2x + 3)^2}$ .

**Solution:** The domain of  $f(x) + \sqrt{9 - x^2}$  is  $[-3, 3]$ . So we have that  $y = f(2x + 3)$  and to find the domain of  $y$ , we apply the linear transformations to  $f$ . So, first we subtract by 3 to get  $[-6, 0]$  and then we divide by 2 to get  $[-3, 0]$  as the domain for  $y$ . The reason we do it in reverse is that you should remember the general mantra that everything you expect would happen, the opposite happens for  $x$ .

42. Find the domain of  $y = \frac{1}{\sqrt{3-x}}$ .

**Solution:** First, the square root must be defined and so  $3 - x \geq 0$  or  $x \leq 3$ . Then, we also need that the denominator cannot be 0 so  $3 - x \neq 0$  or  $x \neq 3$ . Therefore, the domain is  $\{x : x \leq 3 \wedge x \neq 3\} = \{x : x < 3\} = (-\infty, 3)$ .

43. Find the domain and range of  $2 - \arccos(3x + 2)$ .

**Solution:** For the domain, the domain of  $\arccos(x)$  is  $[-1, 1]$ . The linear shift  $3x + 2$  shifts the domain left by 2 then divides by 3 to get  $[-3, -1]$  and then  $[-1, -1/3]$ , which is the domain. For the range, the range of  $\arccos$  is  $[0, \pi]$  and so  $2 - \arccos(3x + 2)$  first multiplies the domain by  $-1$  to get  $[-\pi, 0]$  and then adds 2 to get  $[2 - \pi, 2]$  as the range.

44. Find the domain of  $\frac{\ln(x+3)}{\sqrt{2-x}}$ .

**Solution:** The numerator must exist and hence  $x + 3 > 0$  or  $x > -3$ . The denominator must exist and hence  $2 - x \geq 0$  or  $x \leq 2$ . Also, the denominator cannot equal 0, so  $x \neq 2$ . The domain is the intersection of all 3 requirements and hence the domain is

$$\{x : x > -3 \wedge x \leq 2 \wedge x \neq 2\} = \{x : -3 < x < 2\} = (-3, 2).$$

45. Find the domain of  $\sqrt{\frac{3-x}{1-x}}$ .

**Solution:** We need that  $\frac{3-x}{1-x} \geq 0$ . If  $x > 1$ , then  $1-x < 0$  and hence the inequality changes direction so multiplying by  $1-x$  gives  $3-x \leq 0$  or  $x \geq 3$ . If  $x < 1$ , then  $1-x > 0$  so the inequality doesn't change direction so  $3-x \geq 0$  or  $x \leq 3$ . Thus, the domain is the union of the regions

$$\{x : x > 1 \wedge x \geq 3\} \cup \{x : x < 1 \wedge x \leq 3\} = (-\infty, 1) \cup [3, \infty).$$

## Inverse Functions

46. Find the inverse of  $f(x) = \frac{-2}{x} - 1$ .

**Solution:** We want to solve for  $x$  in terms of  $y$  so

$$x(y+1) = -2 \implies x = \frac{-2}{y+1}$$

is the inverse.

47. Find the inverse of  $\frac{4+\sqrt{3x}}{5}$ .

**Solution:** Solving for  $x$  in terms of  $x$  gives  $\sqrt{3x} = 5y - 4$  so  $x = (5y - 4)^2/3$ .

48. Find the inverse to  $x^2$  on  $(-\infty, 0]$ .

**Solution:** The inverse to  $x^2$  is  $\pm\sqrt{x}$  but since the domain of  $x^2$  is  $(-\infty, 0]$ , the range of the inverse should be  $(-\infty, 0]$  which means that we should take the negative sign so the inverse is  $-\sqrt{x}$ .

49. Find the inverse to  $e^{2x+3}$ .

**Solution:** The inverse has  $\ln y = 2x + 3$  so solving for  $x$  gives  $x = \frac{-3 + \ln y}{2}$ .

50. Find the inverse to  $-\sqrt{\ln x}$ .

**Solution:** Solving for  $x$  in terms of  $y$  gives  $x = e^{y^2}$ . But, the range of  $x$  is  $(-\infty, 0]$  and hence the domain of the inverse is  $(-\infty, 0]$ . So the inverse is  $e^{-y^2}$  on  $(-\infty, 0]$ .

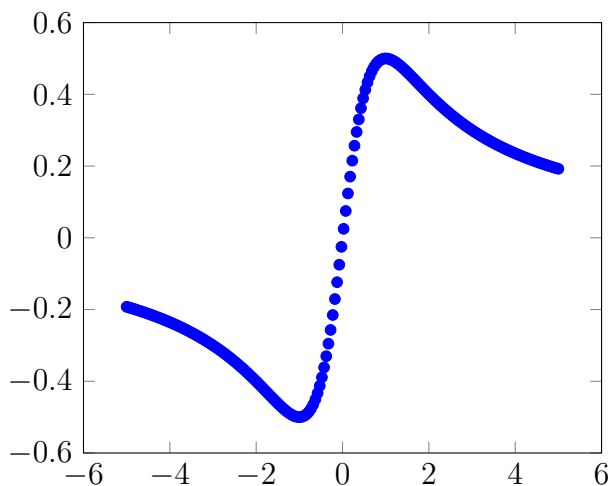
## Graphing

51. Sketch the graph of  $f(x) = \frac{x}{x^2 + 1}$ .

**Solution:** Take the derivative and second derivative to get  $f'(x) = \frac{-(x^2 - 1)}{(x^2 + 1)^2}$  and  $f''(x) = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$ . We want to make a table and the values we care about are when  $f'(x) = 0$ ,  $f''(x) = 0$ , and when they are not defined. They are always defined and solving  $f'(x) = 0$  gives  $x^2 - 1 = 0$  so  $x = \pm 1$ , and  $f''(x) = 0$  gives  $x(x^2 - 3) = 0$ . So the points we need to put in our table are  $x = 0, \pm 1, \pm\sqrt{3}$ . We fill out the table the sign of  $f', f''$  on these intervals to get

	$< -\sqrt{3}$	$-\sqrt{3}$		$-1$		$0$		$1$		$\sqrt{3}$	$\sqrt{3} <$
$f'(x)$	-	-	-	0	+	+	+	0	-	-	-
$f''(x)$	-	0	+	+	+	0	-	-	-	0	+

Now we calculate the limits as  $x \rightarrow \pm\infty$ . We have  $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 0$ . So there is a horizontal asymptote at  $y = 0$ . We can now use this to produce something similar to the following graph noting that  $f$  will have a local minimum at  $x = -1$  and maximum at  $x = 1$  by the second derivative test.



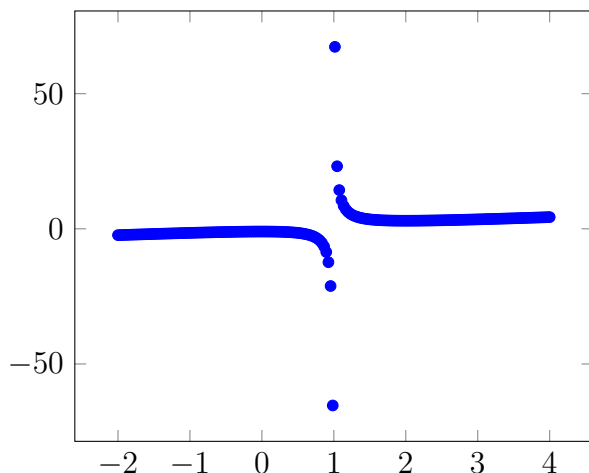
52. Sketch the graph of  $f(x) = x + \frac{1}{x-1}$ .



**Solution:** Take the derivative and second derivative to get  $f'(x) = 1 - \frac{1}{(x-1)^2}$  and  $f''(x) = \frac{2}{(x-1)^3}$ . We want to make a table and the values we care about are when  $f'(x) = 0$ ,  $f''(x) = 0$ , and when they are not defined. They are not defined when  $x = 1$  and solving  $f'(x) = 0$  gives  $(x-1)^2 = 1$  so  $x = 0, 2$ , and  $f''(x) = 0$  has no solutions. So the points we need to put in our table are  $x = 0, 1, 2$ . We fill out the table the sign of  $f', f''$  on these intervals to get

	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, 2)$	2	$(2, \infty)$
$f'(x)$	+	0	-	DNE	-	0	+
$f''(x)$	-	-	-	DNE	+	+	+

Now we calculate the limits as  $x \rightarrow \pm\infty$ . We have  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ ,  $\lim_{x \rightarrow \infty} f(x) = \infty$ . Then since  $f$  is not defined at  $x = 1$ , we calculate the limits of  $f$  there with  $\lim_{x \rightarrow 1^-} f(x) = -\infty$ ,  $\lim_{x \rightarrow 1^+} f(x) = \infty$ . So there is a vertical asymptote at  $x = 1$ . We can now use this to produce something similar to the following graph noting that  $f$  will have a local minimum at  $x = 0$  and maximum at  $x = 2$  by the second derivative test.

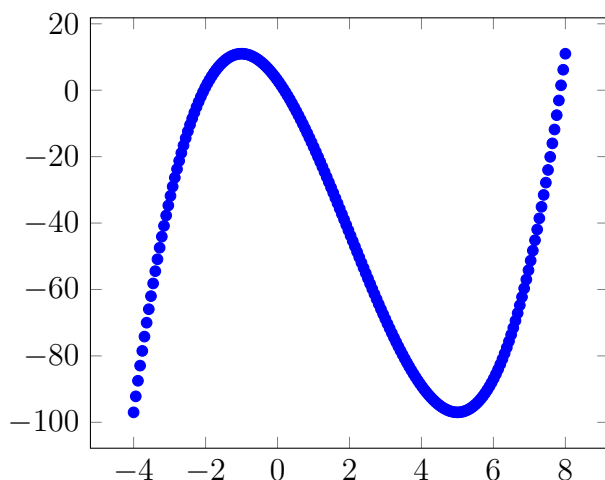


53. Sketch the graph of  $f(x) = 3 - 15x - 6x^2 + x^3$ .

**Solution:** Take the derivative and second derivative to get  $f'(x) = -15 - 12x + 3x^2$  and  $f''(x) = 6x - 12$ . We want to make a table and the values we care about are when  $f'(x) = 0$ ,  $f''(x) = 0$ , and when they are not defined. They are always defined and solving  $f'(x) = 0$  gives  $x^2 - 4x - 5 = 0$  so  $x = -1, 5$ , and  $f''(x) = 0$  gives  $x = 2$ . So the points we need to put in our table are  $x = -1, 2, 5$ . We fill out the table the sign of  $f', f''$  on these intervals to get

	$(-\infty, -1)$	-1	$(-1, 2)$	2	$(2, 5)$	5	$(5, \infty)$
$f'(x)$	+	0	-	-	-	0	+
$f''(x)$	-	-	-	0	+	+	+

Now we calculate the limits as  $x \rightarrow \pm\infty$ . We have  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ ,  $\lim_{x \rightarrow \infty} f(x) = \infty$ . We can now use this to produce something similar to the following graph noting that  $f$  will have a local minimum at  $x = 5$  and maximum at  $x = -1$  by the second derivative test.

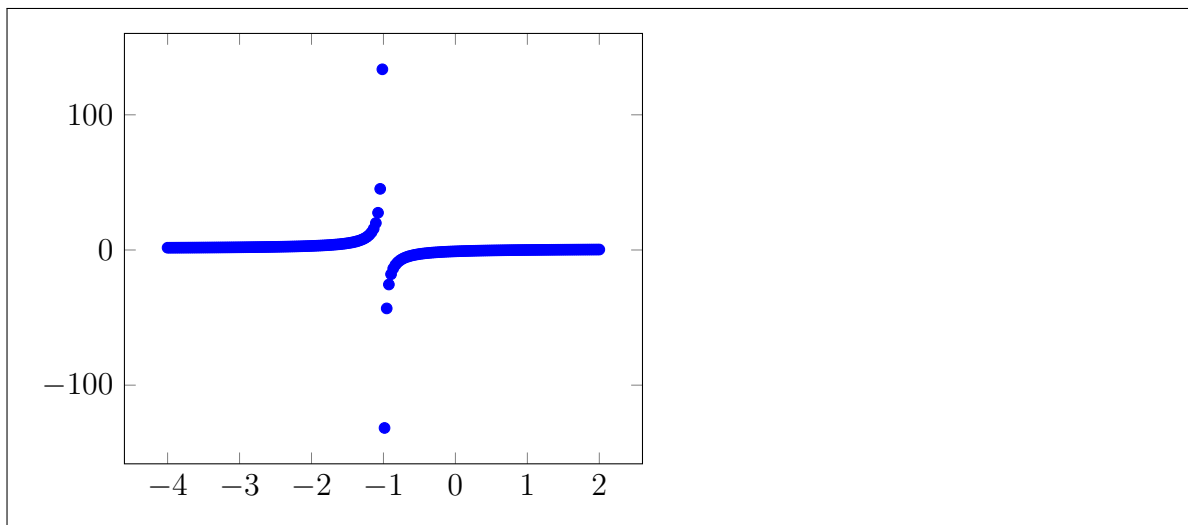


54. Sketch the graph of  $f(x) = \frac{x-1}{x+1}$ .

**Solution:** Take the derivative and second derivative to get  $f'(x) = \frac{2}{(x+1)^2}$  and  $f''(x) = \frac{-4}{(x+1)^3}$ . We want to make a table and the values we care about are when  $f'(x) = 0$ ,  $f''(x) = 0$ , and when they are not defined. They are not defined at  $x = -1$  and solving  $f'(x) = 0$ ,  $f''(x) = 0$  has no solutions. So the points we need to put in our table is just  $x = -1$ . We fill out the table the sign of  $f'$ ,  $f''$  on these intervals to get

	$(-\infty, -1)$	$-1$	$(-1, \infty)$
$f'(x)$	+	DNE	+
$f''(x)$	+	DNE	+

Now we calculate the limits as  $x \rightarrow \pm\infty$ . We have  $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = 1$ . So there is a horizontal asymptote at  $y = 1$ . Now we calculate what happens as  $x \rightarrow -1$  and we have  $\lim_{x \rightarrow -1^-} f(x) = \infty$ ,  $\lim_{x \rightarrow -1^+} f(x) = -\infty$  so it has a vertical asymptote at  $x = -1$ . We can now use this to produce something similar to the following graph.

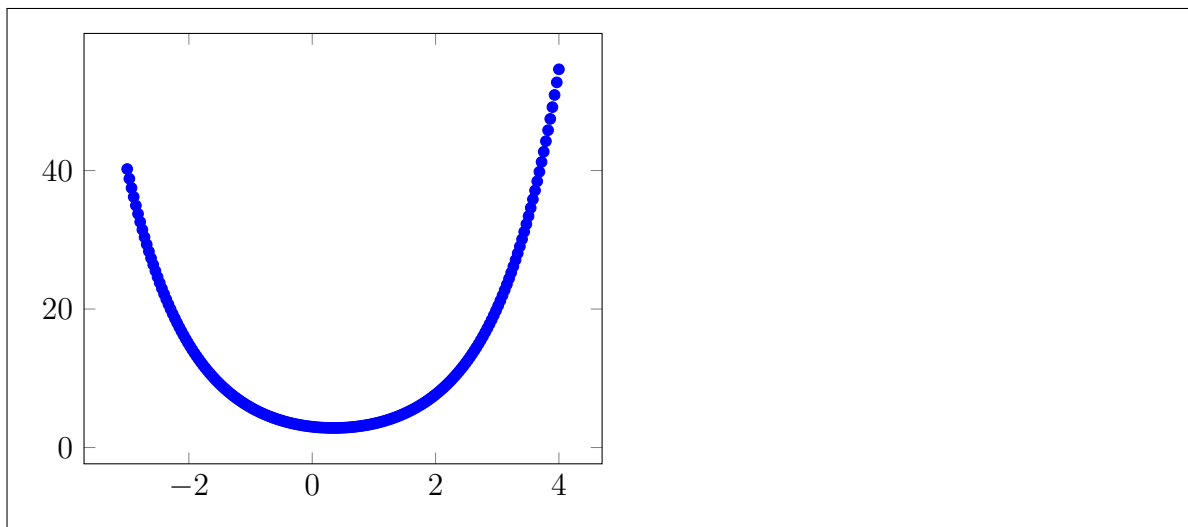


55. Sketch the graph of  $f(x) = e^x + 2e^{-x}$ .

**Solution:** Take the derivative and second derivative to get  $f'(x) = e^x - 2e^{-x}$  and  $f''(x) = e^x + 2e^{-x}$ . We want to make a table and the values we care about are when  $f'(x) = 0$ ,  $f''(x) = 0$ , and when they are not defined. They are always defined and solving  $f'(x) = 0$  gives  $e^{2x} = 2$  so  $x = \frac{\ln 2}{2}$ , and  $f''(x) = 0$  has no solutions. So the point we need to put in is just  $x = \frac{\ln 2}{2}$ . We fill out the table the sign of  $f'$ ,  $f''$  on these intervals to get

	$(-\infty, (\ln 2)/2)$	$(\ln 2)/2$	$((\ln 2)/2, \infty)$
$f'(x)$	-	0	+
$f''(x)$	+	+	+

Now we calculate the limits as  $x \rightarrow \pm\infty$ . We have  $\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow \infty} f(x) = \infty$ . We can now use this to produce something similar to the following graph noting that  $f$  will have a local minimum at  $x = \frac{\ln 2}{2}$  by the second derivative test.



## 1 True/False

56. **TRUE** False You can tell what the domain and range of an inverse function is only from the domain and range of the original function
57. **TRUE** False The horizontal line test tells us whether a function is injective or surjective.
58. True **FALSE** The range of  $e^x$  is  $[0, \infty)$ .
59. True **FALSE** If  $\lim_{x \rightarrow 0} f(x)$  exists, then  $\lim_{x \rightarrow 0} f(x) = f(0)$ .
60. **TRUE** False Extrema of a function must occur when the derivative is 0, when it doesn't exist, or at the endpoints.
61. True **FALSE** The expression  $0^\infty$  is an indeterminate.
62. **TRUE** False The continuity law for subtraction follows from the limit law for subtraction.
63. **TRUE** False The continuity of a constant function follows from limit laws.
64. True **FALSE** The continuity law for rational functions follows only from the limit laws for ratios.
65. **TRUE** False If  $f$  is not continuous at  $x = c$ , then  $f$  is not differentiable at  $x = c$ .
66. True **FALSE** We can use the power rule to find the derivative of  $x^x$ .